CHAPTER 6

INTEGRATION

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Define integration.
2. Find the area under a curve and interpret indefinite integrals.
3. Apply rules of integration.
4. Apply definite integrals to problem solving.

INTRODUCTION

The two main branches of calculus are differential calculus and integral calculus. Having studied differential calculus in previous chapters, we now turn our attention to integral calculus. Basically, integration is the inverse of differentiation just as division is the inverse of multiplication, and as subtraction is the inverse of addition.

DEFINITIONS

Integration is defined as the inverse of differentiation. When we were dealing with differentiation, we were given a function $F(x)$ and were required to find the derivative of this function. In integration we will be given the derivative of a function and will be required to find the function. That is, when we are given the function $f(x)$, we will find another function, $F(x)$, such that

$$\frac{d\,F(x)}{dx} = f(x) \qquad (6.1)$$

In other words, when we have the function $f(x)$, we must find the function $F(x)$ whose derivative is the function $f(x)$.

If we change equation (6.1) to read

$$d\,F(x) = f(x)\,dx \tag{6.2}$$

we have used dx as a *differential*. An equivalent statement for equation (6.2) is

$$F(x) = \int f(x)\,dx$$

We call $f(x)$ the *integrand*, and we say $F(x)$ is equal to the indefinite integral of $f(x)$. The elongated $S, \int$, is the *integral sign*. This symbol is used because integration may be shown to be the limit of a sum.

INTERPRETATION OF AN INTEGRAL

We will use the area under a curve for the interpretation of an integral. You should realize, however, that an integral may represent many things, and it may be real or abstract. It may represent the plane area, volume, or surface area of some figure.

AREA UNDER A CURVE

To find the area under a curve, we must agree on what is desired. In figure 6-1, where $f(x)$ is equal to the constant 4 and the "curve" is the straight line

$$y = 4$$

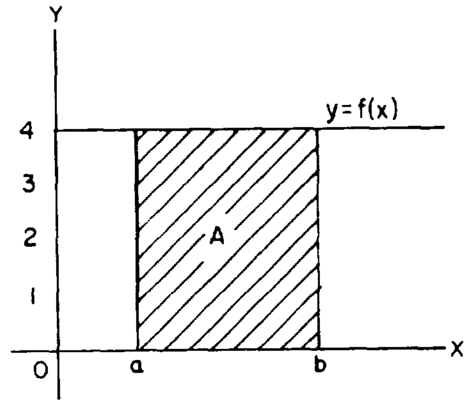

Figure 6-1.—Area of a rectangle.

the area of the rectangle is found by multiplying the height times the width. Thus, the area under the curve is

$$A = 4(b - a)$$

The next problem is to find a method for determining the area under any curve, provided that the curve is continuous. In figure 6-2, the area under the curve

$$y = f(x)$$

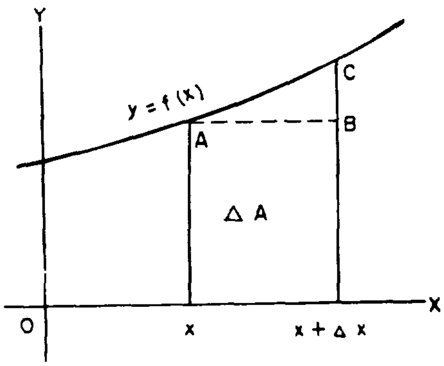

Figure 6-2.—Area ΔA.

between points x and $x + \Delta x$ is approximately $f(x)\Delta x$. We consider that Δx is small and the area given is ΔA.

This area under the curve is nearly a rectangle. The area ΔA, under the curve, would differ from the area of the rectangle by the area of the triangle ABC if AC were a straight line.

When Δx becomes smaller and smaller, the area of ABC becomes smaller at a faster rate, and ABC finally becomes indistinguishable from a triangle. The area of this triangle becomes negilible when Δx is sufficiently small. Therefore, for sufficiently small values of Δx, we can say that

$$\Delta A \approx f(x)\Delta x$$

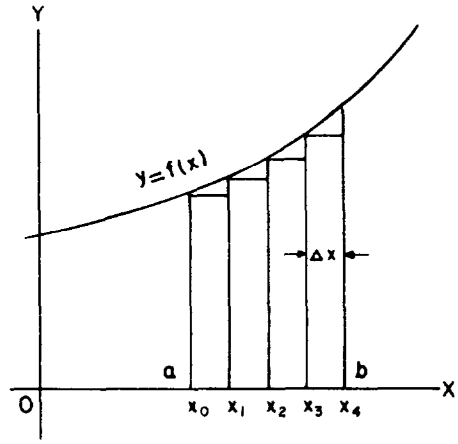

Figure 6-3.—Area of strips.

Now, if we have the curve in figure 6-3, the sum of all the rectangles will be approximately equal to the area under the curve and bounded by the lines at a and b. The difference between the actual area under the curve and the sum of the areas of the rectangles will be the sum of the areas of the triangles above each rectangle.

As Δx is made smaller and smaller, the sum of the rectangular areas will approach the value of the area under the curve. The sum of the areas of the rectangles may be indicated by

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x \qquad (6.3)$$

where Σ (sigma) is the symbol for sum, n is the number of rectangles, $f(x)\Delta x$ is the area of each rectangle, and k is the designation number of each rectangle. In the particular example just discussed, where we have four rectangles, we would write

$$A = \sum_{k=1}^{4} f(x_k)\Delta x$$

and we would have only the sum of four rectangles and not the limiting area under the curve.

When using the limit of a sum, as in equation (6.3), we are required to use extensive algebraic techniques to find the actual area under the curve.

To this point we have been given a choice of using arithmetic and finding only an approximation of the area under a curve or using extensive algebra to find the actual area.

We will now use calculus to find the area under a curve fairly easily.

In figure 6-4, the areas under the curve, from a to b, is shown as the sum of the areas of ${}_aA_c$ and ${}_cA_b$. The notation ${}_aA_c$ means the area under the curve from a to c.

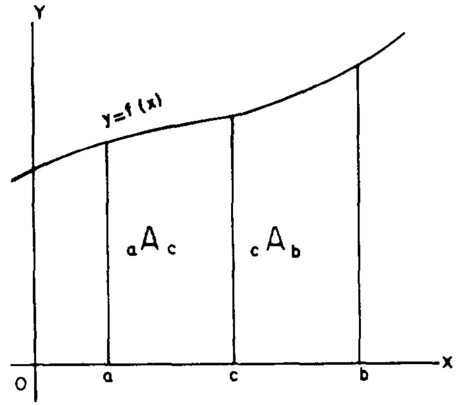

Figure 6-4.—Designation of limits.

The *Intermediate Value Theorem* states that

$$ {}_aA_b = f(c)\ (b - a) $$

where $f(c)$ in figure 6-4 is the value of the function at an intermediate point between a and b.

We now modify figure 6-4 as shown in figure 6-5.

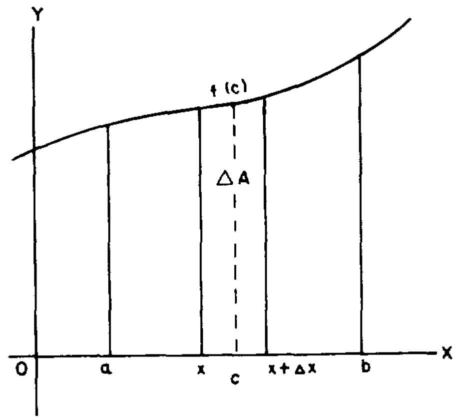

Figure 6-5.—Increments of area at $f(c)$.

When

$$ x = a $$

then

$$ {}_aA_a = 0 $$

We see in figure 6-5 that

$$ {}_aA_x + {}_xA_{(x+\Delta x)} = {}_aA_{(x+\Delta x)} $$

therefore, the increase in area, as shown, is

$$ \Delta A = {}_xA_{(x+\Delta x)} $$

Reference to figure 6-5 shows

$$ {}_xA_{(x+\Delta x)} = f(c)\ (x + \Delta x - x) = f(c)\Delta x $$

where c is a point between a and b. Then by substitution

$$ \Delta A = f(c)\Delta x $$

or

$$ \frac{\Delta A}{\Delta x} = f(c) $$

and as Δx approaches zero, we have

$$\frac{d(A)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x}$$

$$= \lim_{c \to x} f(c)$$

$$= f(x)$$

Now, from the definition of integration

$$_aA_x = \int f(x)\, dx \qquad (6.4)$$
$$= F(x) + C$$

where C is the constant of integration, and

$$_aA_a = F(a) + C$$

but

$$_aA_a = 0$$

therefore,

$$F(a) + C = 0$$

By solving for C, we have

$$C = -F(a)$$

and by substituting $-F(a)$ into equation (6.4), we find

$$_aA_x = F(x) - F(a)$$

If we let

$$x = b$$

then

$$_aA_b = F(b) - F(a) \qquad (6.5)$$

where $F(b)$ and $F(a)$ are the integrals of the function of the curve at the values b and a.

The constant of integration C is omitted in equation (6.5) because when the function of the curve at b and a is integrated, C will occur with both $F(a)$ and $F(b)$ and will therefore be subtracted from itself.

NOTE: The concept of the constant of integration is more fully explained later in this chapter.

EXAMPLE: Find the area under the curve

$$y = 2x - 1$$

in figure 6-6, bounded by the vertical lines at a and b and the X axis.

SOLUTION: We know that

$$_aA_b = F(b) - F(a)$$

and we find that

$$F(x) = \int f(x)\,dx$$

$$= \int (2x - 1)\,dx$$

$$= x^2 - x \text{ (This step will be justified later.)}$$

Then, substituting the values for a and b into $F(x) = x^2 - x$, we find that when

$$x = a$$

$$= 1$$

$$F(a) = 1 - 1$$

$$= 0$$

and when

$$x = b$$

$$= 5$$

$$F(b) = 25 - 5$$

$$= 20$$

Then by substituting these values in

$$_aA_b = F(b) - F(a)$$

we find that

$$_aA_b = 20 - 0$$

$$= 20$$

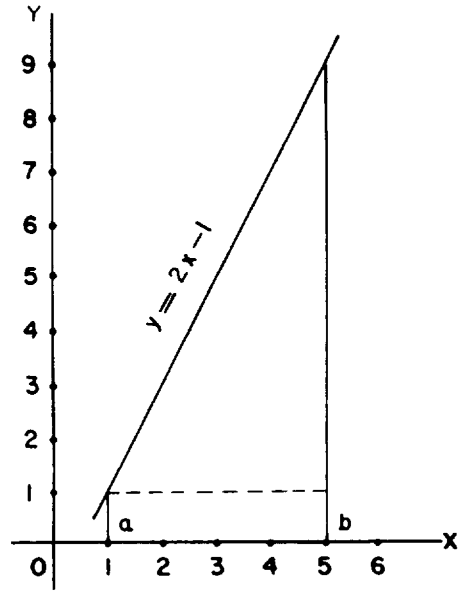

Figure 6-6.—Area of triangle and rectangle.

We may verify this by considering figure 6-6 to be a triangle with base 4 and height 8 sitting on a rectangle of height 1 and base 4. By known formulas, we find the area under the curve to be 20.

INDEFINITE INTEGRALS

When we were finding the derivative of a function, we wrote

$$\frac{d\,F(x)}{dx} = f(x)$$

where the derivative of $F(x)$ is $f(x)$. Our problem is to find $F(x)$ when we are given $f(x)$.

We know that the symbol $\int \ldots dx$ is the inverse of $\frac{d}{dx}$, or when dealing with differentials, the operator symbols d and $\int$ are the inverse of each other; that is,

$$F(x) = \int f(x)\, dx$$

and when the derivative of each side is taken, d annulling $\int$, we have

$$d\, F(x) = f(x)\, dx$$

or where $\int \ldots dx$ annuals $\frac{d}{dx}$, we have

$$\frac{d\,F(x)}{dx} = \frac{d}{dx}\int f(x)\, dx$$

$$= f(x)$$

From this, we find that

$$d\,(x^3) = 3x^2\, dx$$

so that,

$$\int 3x^2\, dx = x^3 + C$$

Also we find that

$$d\,(x^3 + 3) = 3x^2\, dx$$

so that,

$$\int 3x^2\, dx = x^3 + 3$$

Again, we find that

$$d\,(x^3 - 9) = 3x^2\, dx$$

so that,

$$\int 3x^2\, dx = x^3 - 9$$

This is to say that

$$d\,(x^3 + C) = 3x^2\, dx$$

and

$$\int 3x^2\, dx = x^3 + C$$

where C is any *constant of integration*.

A number that is independent of the variable of integration is called a *constant of integration*. Since C may have infinitely many values, then a differential expression may have infinitely many integrals differing only by the constant. This is to say that two integrals of the same function may differ by the constant of integration. We assume the differential expression has at least one integral. Because the integral contains C and C is indefinite, we call

$$F(x) + C$$

an *indefinite integral* of $f(x)\, dx$. In the general form we say

$$\int f(x)\, dx = F(x) + C$$

With regard to the constant of integration, a theorem and its converse are stated as follows:

Theorem 1. *If two functions differ by a constant, they have the same derivative.*

Theorem 2. *If two functions have the same derivative, their difference is a constant.*

RULES FOR INTEGRATION

Although integration is the inverse of differentiation and we were given rules for differentiation, we are required to determine the answers in integration by trial and error. However, there are some rules to aid us in the determination of the answer.

In this section we will discuss four of these rules and how they are used to integrate standard elementary forms. In the rules we will let u and v denote a differentiable function of a variable such as x. We will let C, n, and a denote constants.

Our proofs will involve searching for a function $F(x)$ whose derivative is $f(x)\,dx$.

Rule 1. $\int du = u + C$

The integral of a differential of a function is the function plus a constant.

PROOF: If

$$\frac{d\,(u + C)}{du} = 1$$

then

$$d\,(u + C) = du$$

and

$$\int du = u + C$$

EXAMPLE: Evaluate the integral

$$\int dx$$

SOLUTION: By Rule 1, we have

$$\int dx = x + C$$

Rule 2. $\int a\,du = a \int du = au + C$

A constant may be moved across the integral sign. NOTE: A variable may NOT be moved across the integral sign.

PROOF: If

$$\frac{d\,(au + C)}{du} = (a)\frac{d\,(u + C)}{du} = a$$

then

$$d\ (au + C) = a\ d\ (u + C) = a\ du$$

and

$$\int a\ du = a \int du = au + C$$

EXAMPLE: Evaluate the integral

$$\int 4\ dx$$

SOLUTION: By Rule 2,

$$\int 4\ dx = 4 \int dx$$

and by Rule 1,

$$\int dx = x + C$$

therefore,

$$\int 4\ dx = 4x + C$$

Rule 3. $\displaystyle\int u^n\ du = \frac{u^{n+1}}{n+1} + C$

The integral of u^n du may be obtained by adding 1 to the exponent and then dividing by this new exponent. NOTE: If n is minus 1, this rule is not valid and another method must be used.

PROOF: If

$$d\left(\frac{u^{n+1}}{n+1} + C\right) = \frac{(n+1)u^n}{n+1}\ du$$

$$= u^n\ du$$

then

$$\int u^n\ du = \frac{u^{n+1}}{n+1} + C$$

EXAMPLE: Evaluate the integral

$$\int x^3\ dx$$

SOLUTION: By Rule 3,

$$\int x^3\, dx = \frac{x^{3+1}}{3+1} + C$$

$$= \frac{x^4}{4} + C$$

EXAMPLE: Evaluate the integral

$$\int \frac{7}{x^3}\, dx$$

SOLUTION: First write the integral

$$\int \frac{7}{x^3}\, dx$$

as

$$\int 7x^{-3}\, dx$$

Then, by Rule 2,

$$7 \int x^{-3}\, dx$$

and by Rule 3,

$$7 \int x^{-3}\, dx = 7\left(\frac{x^{-2}}{-2}\right) + C = -\frac{7}{2x^2} + C$$

Rule 4. $\int (du + dv + dw) = \int du + \int dv + \int dw$

$$= u + v + w + C$$

The integral of a sum is equal to the sum of the integrals.

PROOF: If

$$d\,(u + v + w + C) = du + dv + dw$$

then

$$\int (du + dv + dw) = (u + C_1) + (v + C_2) + (w + C_3)$$

such that

$$\int (du + dv + dw) = u + v + w + C$$

where

$$C = C_1 + C_2 + C_3$$

EXAMPLE: Evaluate the integral

$$\int (2x - 5x + 4)\, dx$$

SOLUTION: We will not combine $2x$ and $-5x$.

$$\int (2x - 5x + 4)\, dx$$

$$= \int 2x\, dx - \int 5x\, dx + \int 4\, dx$$

$$= 2 \int x\, dx - 5 \int x\, dx + 4 \int dx$$

$$= \frac{2x^2}{2} + C_1 - \frac{5x^2}{2} + C_2 + 4x + C_3$$

$$= x^2 - \frac{5}{2} x^2 + 4x + C$$

where C is the sum of C_1, C_2, and C_3.

EXAMPLE: Evaluate the integral

$$\int (x^{1/2} + x^{2/3})\, dx$$

SOLUTION:

$$\int (x^{1/2} + x^{2/3})\, dx$$

$$= \int x^{1/2}\, dx + \int x^{2/3}\, dx$$

$$= \frac{x^{3/2}}{3/2} + C_1 + \frac{x^{5/3}}{5/3} + C_2$$

$$= \frac{2x^{3/2}}{3} + \frac{3x^{5/3}}{5} + C$$

Now we will discuss the evaluation of the constant of integration.

If we are to find the equation of a curve whose first derivative is 2 times the independent variable x, we may write

$$\frac{dy}{dx} = 2x$$

or

$$dy = 2x\, dx \qquad (1)$$

We may obtain the desired equation for the curve by integrating the expression for dy; that is, by integrating both sides of equation (1). If

$$dy = 2x\,dx$$

then,

$$\int dy = \int 2x\,dx$$

But, since

$$\int dy = y$$

and

$$\int 2x\,dx = x^2 + C$$

then

$$y = x^2 + C$$

We have obtained only a general equation of the curve because a different curve results for each value we assign to C. This is shown in figure 6-7. If we specify that

$$x = 0$$

and

$$y = 6$$

we may obtain a specific value for C and hence a particular curve.

Suppose that

$$y = x^2 + C,\ x = 0, \text{ and } y = 6$$

then,

$$6 = 0^2 + C$$

or

$$C = 6$$

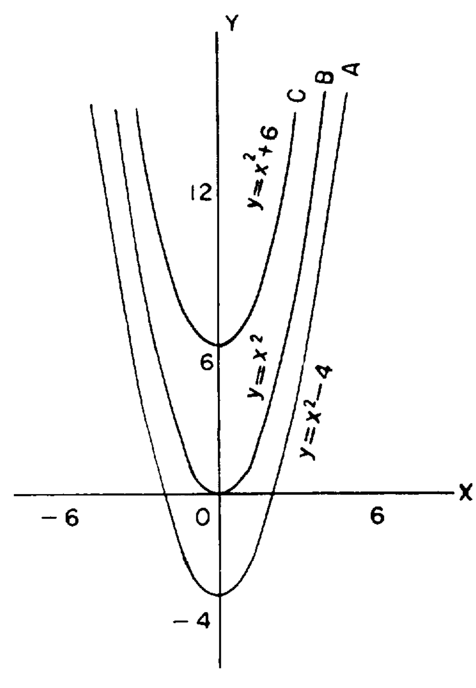

Figure 6-7.—Family of curves.

By substituting the value 6 into the general equation, we find that the equation for the particular curve is

$$y = x^2 + 6$$

which is curve C of figure 6-7.

The values for x and y will determine the value for C and also determine the particular curve of the family of curves.

In figure 6-7, curve A has a constant equal to -4, curve B has a constant equal to 0, and curve C has a constant equal to 6.

EXAMPLE: Find the equation of the curve if its first derivative is 6 times the independent variable, y equals 2, and x equals 0.

SOLUTION: We may write

$$\frac{dy}{dx} = 6x$$

or

$$\int dy = \int 6x\, dx$$

such that,

$$y = 3x^2 + C$$

Solving for C when

$$x = 0$$

and

$$y = 2$$

we have

$$2 = 3(0^2) + C$$

or

$$C = 2$$

so that the equation of the curve is

$$y = 3x^2 + 2$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int x^2\, dx$

2. $\int 4x\, dx$

3. $\int (x^3 + x^2 + x)\, dx$

4. $\int 6\, dx$

5. $\int \frac{5}{x^2}\, dx$

6. Find the equation of the curve if its first derivative is 9 times the independent variable squared, y equals 5, and x equals -1.

ANSWERS:

1. $\frac{x^3}{3} + C$

2. $2x^2 + C$

3. $\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + C$

4. $6x + C$

5. $-\frac{5}{x} + C$

6. $y = 3x^3 + 8$

DEFINITE INTEGRALS

The general form of the indefinite integral is

$$\int f(x)\, dx = F(x) + C$$

and has two identifying characteristics. First, the constant of integration must be added to each integration. Second, the result of integration is a function of a variable and has no definite value, even after the constant of integration is determined, until the variable is asigned a numerical value.

The definite integral eliminates these two characteristics. The form of the *definite integral* is

$$\int_a^b f(x)\,dx = [F(b) + C] - [F(a) + C] \tag{6.6}$$

$$= F(b) - F(a)$$

where a and b are given values. Notice that the constant of integration does not appear in the final expression of equation (6.6). In words, this equation states that the difference of the values of

$$\int_a^b f(x)\,dx$$

for

$$x = a$$

and

$$x = b$$

gives the area under the curve defined by $f(x)$, the X axis, and the ordinates where

$$x = a$$

and

$$x = b$$

In figure 6-8, the value of b is the upper limit and the value of a is the lower limit. These upper and lower limits may be any assigned values in the range of the curve. The upper limit is positive with respect to the lower limit in that it is located to the right (positive in our case) of the lower limit.

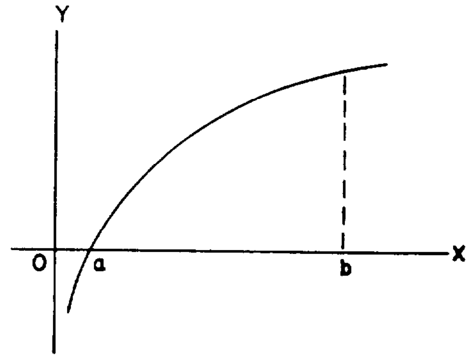

Figure 6-8.—Upper and lower limits.

Equation (6.6) is the limit of the sum of all the strips between a and b, having areas of $f(x)\Delta x$; that is,

$$\lim \sum_{x=a}^{x=b} f(x)\Delta x = \int_a^b f(x)\,dx$$

The definite integral evaluated from a to b is

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b \qquad (6.7)$$

$$= F(b) - F(a)$$

The notation $F(x)\Big|_a^b$ in equation (6.7) means we first substitute the upper limit, b, into the function $F(x)$ to obtain $F(b)$; and from $F(b)$ we subtract $F(a)$, the value obtained by substituting the lower limit, a, into $F(x)$.

EXAMPLE: Find the area bounded by the curve

$$y = x^2$$

the X axis, and the ordinates where

$$x = 2$$

and

$$x = 3$$

as shown in figure 6-9.

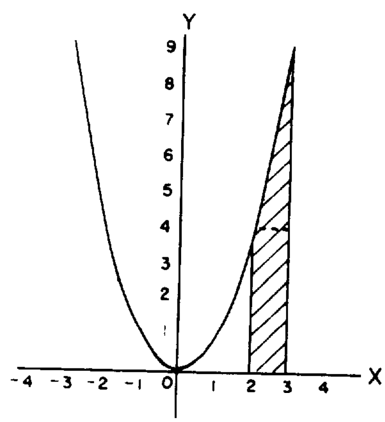

Figure 6-9.—Area from $x = 2$ to $x = 3$.

SOLUTION: Substituting into equation (6.7), we have

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

$$= \int_2^3 x^2\,dx$$

$$= \frac{x^3}{3}\Big|_2^3$$

$$= \frac{3^3}{3} - \frac{2^3}{3}$$

$$= \frac{27}{3} - \frac{8}{3}$$

$$= \frac{19}{3}$$

$$= 6\,\frac{1}{3}$$

We may make an estimate of this solution by considering the area desired in figure 6-9 as being a right triangle resting on a rectangle. The triangle has an approximate area of

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}(1)(5)$$

$$= \frac{5}{2}$$

and the area of the rectangle is

$$A = bh$$

$$= (1)(4)$$

$$= 4$$

so that the total area is

$$4 + \frac{5}{2} = \frac{13}{2} = 6\,\frac{1}{2}$$

which is a close approximation of the area found by the process of integration.

EXAMPLE: Find the area bounded by the curve

$$y = x^2$$

the X axis, and the ordinates where

$$x = -2$$

and

$$x = 2$$

as shown in figure 6-10.

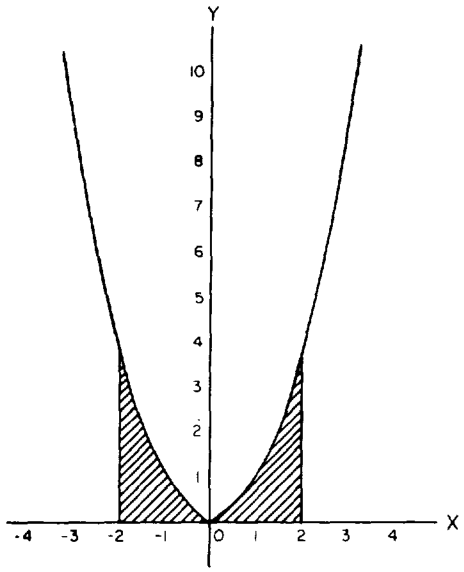

Figure 6-10.—Area under a curve.

SOLUTION: Substituting into equation (6.7), we have

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

$$= F(b) - F(a)$$

$$= \int_{-2}^{2} x^2\,dx$$

$$= \frac{x^3}{3}\Big|_{-2}^{2}$$

$$= \frac{8}{3} - \left(-\frac{8}{3}\right)$$

$$= \frac{16}{3}$$

$$= 5\,\frac{1}{3}$$

The area above a curve and below the X axis, as shown in figure 6-11, will, through integration, furnish a negatvie answer.

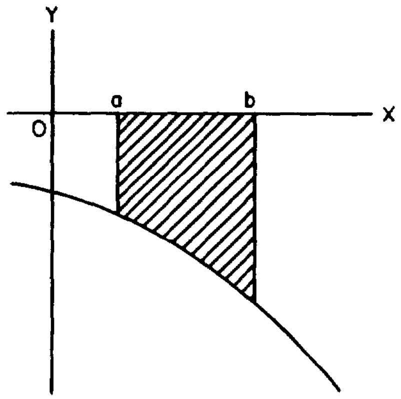

Figure 6-11.—Area above a curve.

If the graph of $y = f(x)$, *between* $x = a$ *and* $x = b$, *has portions above and portions below the X axis,* as shown in figure 6-12, *then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

is the sum of the absolute values of the positive areas above the X axis and the negative areas below the X axis, such that

$$\int_a^b f(x)\,dx = |A_1| + |A_2| + |A_3| + |A_4|$$

where

$$A_1 = \int_a^c f(x)\,dx \qquad A_2 = \int_c^d f(x)\,dx$$

$$A_3 = \int_d^e f(x)\,dx \qquad A_4 = \int_e^b f(x)\,dx$$

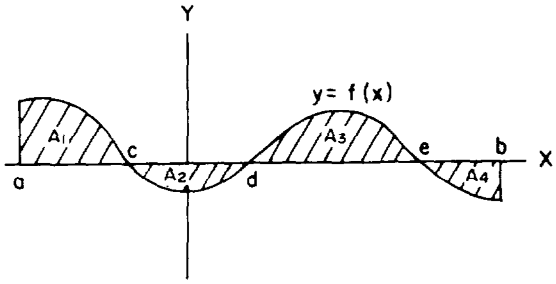

Figure 6-12.—Areas above and below a curve.

EXAMPLE: Find the areas between the curve

$$y = x$$

and the X axis bounded by the lines

$$x = -2$$

and

$$x = 2$$

as shown in figure 6-13.

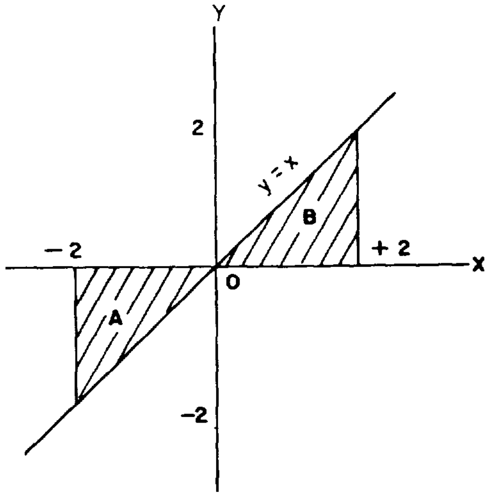

Figure 6-13.—Negative and positive value areas.

SOLUTION: These areas must be computed separately; therefore, we write

$$\begin{aligned} \text{Area A} &= \int_{-2}^{0} f(x)\,dx \\ &= \int_{-2}^{0} x\,dx \\ &= \left.\frac{x^2}{2}\right|_{-2}^{0} \\ &= 0 - \frac{4}{2} \\ &= -2 \end{aligned}$$

and the absolute value of -2 is

$$|-2| = 2$$

Then,

$$\text{Area } B = \int_0^2 f(x)\,dx$$

$$= \frac{x^2}{2}\Bigg|_0^2$$

$$= \frac{4}{2} - 0$$

$$= 2$$

Adding the two areas, $|A|$ and $|B|$, we find

$$|A| + |B| = 2 + 2$$

$$= 4$$

NOTE: If the function is integrated from -2 to 2, the following INCORRECT result will occur:

$$\text{Area} = \int_{-2}^{2} f(x)\,dx$$

$$= \int_{-2}^{2} x\,dx$$

$$= \frac{x^2}{2}\Bigg|_{-2}^{2}$$

$$= \frac{4}{2} - \frac{4}{2}$$

$$= 0 \qquad \text{(INCORRECT SOLUTION)}$$

This is obviously not the area shown in figure 6-13. Such an example emphasizes the value of making a commonsense check on every solution. A sketch of the function will aid this commonsense judgement.

EXAMPLE: Find the total area bounded by the curve

$$y = x^3 - 3x^2 - 6x + 8$$

the X axis, and the lines

$$x = -2$$

and

$$x = 4$$

as shown in figure 6-14.

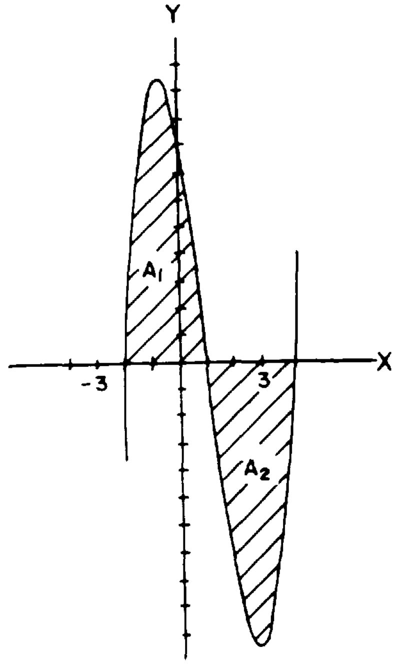

Figure 6-14.—Positive and negative value areas.

SOLUTION: The area desired is both above and below the X axis; therefore, we need to find the areas separately and then add them together using their absolute values.

Therefore,

$$A_1 = \int_{-2}^{1} (x^3 - 3x^2 - 6x + 8)\, dx$$

$$= \frac{x^4}{4} - x^3 - 3x^2 + 8x \Big|_{-2}^{1}$$

$$= \left(\frac{1}{4} - 1 - 3 + 8\right) - \left(4 + 8 - 12 - 16\right)$$

$$= 4\frac{1}{4} + 16$$

$$= 20\frac{1}{4}$$

and

$$A_2 = \int_1^4 (x^3 - 3x^2 - 6x + 8)\, dx$$

$$= \frac{x^4}{4} - x^3 - 3x^2 + 8x \Big|_1^4$$

$$= \left(64 - 64 - 48 + 32\right) - \left(\frac{1}{4} - 1 - 3 + 8\right)$$

$$= -16 - 4\frac{1}{4}$$

$$= -20\frac{1}{4}$$

Then, the total area is

$$|A_1| + |A_2| = 20\frac{1}{4} + 20\frac{1}{4}$$

$$= 40\frac{1}{2}$$

PRACTICE PROBLEMS:

1. Find, by integration, the area under the curve

$$y = x + 4$$

bounded by the X axis and the lines

$$x = 2$$

and

$$x = 7$$

verify this by a geometric process.

2. Find the area under the curve

$$y = 3x^2 + 2$$

bounded by the X axis and the lines

$$x = 0$$

and

$$x = 2$$

3. Find the area between the curve

$$y = x^3 - 12x$$

and the X axis, from

$$x = -1$$

to

$$x = 3$$

ANSWERS:

1. 42 1/2
2. 12
3. 39 1/2

SUMMARY

The following are the major topics covered in this chapter:

1. **Definition of integration:** *Integration* is defined as the inverse of differentiation.

$$F(x) = \int f(x)\,dx$$

where $F(x)$ is the function whose derivative is the function $f(x)$; $\int$ is the *integral sign*; $f(x)$ is the *integrand*; and dx is the *differential.*

2. **Area under a curve:**

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

where Σ (sigma) is the symbol for sum, n is the number of rectangles, $f(x)\Delta x$ is the area of each rectangle, and k is the designation number of each rectangle.

Intermediate Value Theorem:

$${}_aA_b = f(c)(b - a)$$

where $f(c)$ is the function at an intermediate point between a and b.

$${}_aA_b = F(b) - F(a)$$

where $F(b) - F(a)$ are the integrals of the function of the curve at the values b and a.

3. **Indefinite integrals:**

$$\int f(x)\,dx = F(x) + C$$

where C is called a *constant of integration*, a number which is independent of the variable of integration.

Theorem 1. *If two functions differ by a constant, they have the same derivative.*

Theorem 2. *If two functions have the same derivative, their difference is a constant.*

4. **Rules for integration:**

Rule 1. $\int du = u + C$

The integral of a differential of a function is the function plus a constant.

Rule 2. $\int a\ du = a \int du = au + C$

A constant may be moved across the integral sign. NOTE: A variable may NOT be moved across the integral sign.

Rule 3. $\displaystyle\int u^n\ du = \frac{u^{n+1}}{n+1} + C$

The integral of u^n du may be obtained by adding 1 to the exponent and then dividing by this new exponent. NOTE: If n is minus 1, this rule is not valid.

Rule 4. $\int (du + dv + dw) = \int du + \int dv + \int dw$

$$= u + v + w + C$$

The integral of a sum is equal to the sum of the integrals.

5. **Definite integrals:**

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where b, the upper limit, and a, the lower limit, are given values.

6. **Areas above and below a curve:**

If the graph of $y = f(x)$, between $x = a$ and $x = b$, has portions above and portions below the X axis, then

$\int_a^b f(x)\ dx = F(b) - F(a)$ is the sum of the absolute values of the positive areas above the X axis and the negative areas below the X axis.

ADDITIONAL PRACTICE PROBLEMS

Evaluate the following integrals:

1. $\int (3/4)\, dx$

2. $\int \frac{-8\, dx}{x^3}$

3. $\int (7x^6 - 5x^{-6} - 10x^4 + 9x^{-4})\, dx$

4. $\int 18x^{5/4}\, dx$

5. Find the equation of the curve if its first derivative is 4 plus 8 times the independent variable plus 12 times the independent variable cubed, y equals 12, and x equals 2.

6. Find the area under the curve $y = 2x + 3$ bounded by the X axis and the lines $x = -3/2$ and $x = 1$.

7. Find the area between the curve $y = x^2 + 4x$ and the X axis, from $x = -4$ to $x = 0$.

8. Find the total area bounded by the curve $y = 2 + x - x^2$, the X axis, and the lines $x = -2$ and $x = 3$. (Hint: Be sure to sketch the graph.)

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. $(3/4)x + C$

2. $4/x^2 + C$

3. $x^7 + x^{-5} - 2x^5 - 3x^{-3} + C$

4. $8x^{9/4} + C$

5. $y = 3x^4 + 4x^2 + 4x - 60$

6. 25/4

7. 10 2/3

8. 49/6